

## On the Euler Characteristics of Random Surfaces

Robert Schrader<sup>1</sup>

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In a Monte Carlo computer experiment, we simulate the Gibbs distribution of nonconnected two-dimensional surfaces isometrically embedded in three-dimensional Euclidean space with fixed boundary and the action given by the area. The simulation involves surfaces built out of plaquettes in a cubical lattice. The foam structure is analyzed in terms of correlations of the local fluctuations in the Euler characteristic and the area. The scaling behavior of the area and the Euler characteristic is discussed by varying the boundary. We show evidence of a phase transition point which is independent of the choice of the boundary. An existence proof is given of the thermodynamic limit for the models considered.

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**KEY WORDS:** Random surfaces; topological fluctuations; Monte Carlo simulation; phase transition; boundary effects.

### 1. INTRODUCTION

Since Polyakov's article,<sup>(1)</sup> there has been a revived interest in the theory of string quantization. Thus Alvarez<sup>(2)</sup> and Friedan<sup>(3)</sup> discuss the continuum aspects of Polyakov's approach. On the other hand, the lattice versions of the theory of random surfaces are considered important in the context of lattice gauge theories; see, e.g., Refs. 4–12. In particular the formal  $N \rightarrow 0$  limit for  $U(N)$  lattice gauge theories leads to the theory of self-avoiding random surfaces<sup>(13,14)</sup> (for a correction see Ref. 15). This is analogous to the formal  $N \rightarrow 0$  limit for  $N$ -component lattice spin systems, which gives the theory of self-avoiding polymers.<sup>(16–18)</sup>

In this note we want to use a cubical lattice to simulate embedded random surfaces by planar surfaces. We use this example to address the

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<sup>1</sup> Institut für theoretische Physik, Freie Universität, Berlin.

problem of the role of topological fluctuations in quantum field theories. The observable we will study is the Euler characteristic. In this lattice formulation, it will be defined on all configurations of our statistical ensemble. This contrasts, for example, with the situation in lattice gauge theory, where it is not possible to associate a Chern class and number to every lattice gauge configuration (see, e.g., Refs. 19–21).

The aim of string quantization in its Euclidean formulation may be described in purely geometrical terms. Consider an arbitrary, but fixed, smooth Jordan curve  $\gamma$  in  $E^N$ , the Euclidean space in  $N \geq 3$  dimensions, and let  $C(\gamma)$  be the family of all isometrically embedded, smooth compact 2-manifolds  $M^2$  with boundary  $\partial M^2 = \gamma$ .  $M^2$  need not be connected, but if it is, it is called a Teichmüller space. One would like to have a theory in which formally the elements  $M^2$  in  $C(\gamma)$  are distributed with relative distribution

$$\exp[-\beta \text{Area}(M^2)] \quad (1.1)$$

Note that the area action corresponds to the cosmological term in the theory of gravity.  $\beta \geq 0$  is a coupling constant with the dimension of an inverse area. Such a theory would therefore be governed by two scale parameters, namely,  $\beta$  and the length  $l(\gamma)$  of  $\gamma$ . If  $C(\gamma)$  contains a (not necessarily unique) minimal surface  $M_{\min}^2(\gamma)$ , the relative distribution may be rewritten as

$$\exp[-\beta E(M^2)] \quad (1.2)$$

with the nonnegative energy function

$$E(M^2) = \text{Area}(M^2) - \text{Area}(M_{\min}^2(\gamma)) \quad (1.3)$$

as the action of this theory. We then may also introduce the dimensionless coupling constant  $\beta'$  by

$$\beta = \beta' \cdot \text{Area}_{\min}(\gamma) \quad (1.4)$$

A discussion of all possible local counterterms that may appear in this theory has been given in Ref. 2. The structure of the possible local terms in the action is our reason for considering manifolds which are not necessarily connected: There is no local interaction to enforce this property.

This may also be the reason for the difficulty in establishing reflection positivity in other models of random surfaces, where connectedness is required (see, e.g., Ref. 15).

Now a nonconnected manifold with boundary may alternatively be thought of as a connected manifold with boundary immersed in a sea of

closed manifolds (= vacuum fluctuations). In the picture of the Goto–Nambu Strings<sup>(22,61)</sup> it corresponds to an open string (= quark–anti-quark) moving in a surrounding sea of closed strings (= gluons). Therefore globally the effect of the boundary should vanish as long as one looks at densities like the energy density. In the models we will discuss, we shall see that this is indeed the case. However, if we subtract the vacuum fluctuations, one should be able to observe the effects of the boundary, and this is what we will do. Now in a sense one expects this approach to be the same as when one considers connected manifolds only. But at least in the concrete models we will discuss, there will be interference effects of the connected manifold with boundary with the surrounding sea of closed manifolds; in other words, this manifold will at least locally polarize the vacuum. Therefore we expect these two alternative descriptions (connected versus unconnected) not to be equivalent.

Now one local counterterm is the Euler characteristic: By Gauss' Theorema Egregium for any compact two-dimensional Riemannian manifold  $M$  there is a 1-form on  $\partial M = \gamma$  (the second fundamental form) and a 2-form on  $M$  (the Gauss–Bonnet density) which when integrated over  $\gamma$  and  $M$ , respectively, and added up give the Euler characteristic. This is an integer, and as a side remark we note that this property continues to hold if the compactness assumption is replaced by the weaker condition that the manifold is complete, the area is finite, and the curvature is bounded.<sup>(24)</sup>

So on  $C(\gamma)$  the Euler characteristic  $\chi(\cdot)$  is well defined and integer valued. Furthermore, if say,  $\partial M$  is a Jordan curve, there is the relation

$$\begin{aligned} \chi(M) &= 2 \cdot \#(\text{connected components of } M) \\ &\quad - 2 \cdot \text{genus of } M - 1 \end{aligned} \tag{1.5}$$

Here the genus of a 2-manifold is the number of its handles, which intuitively is the number of times a manifold connects up to itself.

Although the connectedness and the genus are also intrinsic (i.e., independent of the particular embedding), they are not locally computable quantities like the Euler characteristic number. Finally there are also invariants under homeomorphism (leaving  $\partial M = \gamma$  fixed) which are not intrinsic. Let, for example,  $\gamma = \emptyset$  and let  $M$  have two components, both of the form of a doughnut. The way they interlock (or not) is not an intrinsic property. In this article we will not be concerned with such extrinsic properties.

Now in the formal probabilistic context given above,  $\chi$  becomes a random variable, and intuitively one expects large fluctuations in topology and in particular in  $\chi$ , since it is possible to deform a given manifold  $M$  into

one with a different topology without using energy. (Note that even if one adds to the action a term proportional to the Euler characteristic, strong fluctuations in the topology will still persist; for example, the same energy arguments continue to apply in the sector  $\chi = 0$ .)

One aim of this paper is to test this intuitive picture numerically in an approximate lattice version. We view this discussion as a step in achieving an understanding of what Wheeler has termed the foamlike structure of space and time and which is expected to occur in a quantized theory of gravity.<sup>(25)</sup>

To obtain a quantitative test of this picture we proceeded as follows. On a lattice we simulated discrete versions of two-dimensional manifolds with a local upgrading by a Monte Carlo procedure. With  $M(t)$  denoting the manifold obtained at computer time  $t$ , we calculated the local differences  $\delta E(x, t)$  and  $\delta \text{Area}(x, t)$  of the Euler characteristic and the area, respectively, of the manifolds  $M(t)$  and  $M(t+1)$ . Let  $\bar{\quad}^t$  denote the time average. If  $\delta \text{Eul}(x, t)$  and  $\delta \text{Area}(x, t)$  were independent random variables with respect to then for any functions  $F$  and  $G$  of  $\delta \text{Eul}(x, t)$  and  $\delta \text{Area}(x, t)$ , respectively, we would have

$$\overline{FG}^t = \bar{F}^t \bar{G}^t \quad (1.6)$$

Hence the closeness of the quantity

$$\overline{FG}^t / \bar{F}^t \bar{G}^t \quad (1.7)$$

(or its inverse) to 1 is a measure of the statistical independence of  $\text{Eul}(x, t)$  and  $\text{Area}(x, t)$ . In our numerical calculations we choose  $F = \delta \text{Eul}(x, t)^2$  and  $G = [\delta \text{Area}(x, t)]^{\pm 2}$ .

$$\frac{\delta \text{Eul}(x, t)}{\delta \text{Area}(x, t)} \quad (1.8)$$

may be interpreted as the local change in the Gauss–Bonnet density of the “manifold”  $M(t)$  at the lattice point  $x$ , where  $x$  is the point where the local upgrading is taking place at time  $t$ .

The lattice version we will use will also allow us to analyze other aspects of random surfaces. For any value of the coupling constant  $\beta$  as already indicated the ensembles  $C(\gamma)$  and  $C(\gamma = \emptyset)$  should only differ locally around  $\gamma$ , i.e., far away from  $\gamma$  a typical representative of  $C(\gamma)$  and one from  $C(\gamma = \emptyset)$  should look the same. In other words, the polarization effect of the boundary  $\gamma$  should extend only to a neighborhood of  $\gamma$ . We test this picture by measuring the difference of both the mean area and Euler characteristic on the configurations  $C(\gamma)$  and  $C(\gamma = \emptyset)$ , respectively.

According to this picture these differences should be intensive quantities. This is supported by our analytical result. Our numerical findings also confirm this. Furthermore they indicate a phase transition with a critical temperature located at  $\beta \sim 0.5$  which is independent of  $\gamma$ . Note that these results do not contradict the fact that for  $\beta$  sufficiently small one expects with probability 1 the connected component of  $M$ , which contains  $\gamma$ , to extend to infinity. The lattice version of this is of course the percolation of surfaces (see, e.g., Refs. 15, 26). The observables we work with, however, are not sensitive enough to measure to this effect.

Finally this system opens up the possibility of discussing loop-loop correlations in the following way. Take  $\gamma = \gamma_1 \cup \gamma_2(x)$ . Here  $\gamma_2(x)$  is a translate by  $x$  of  $\gamma_2$  for  $x \rightarrow \infty$ , one should expect cluster properties. This could be of interest in the context of a study of correlations of Wilson loops. Since larger lattices are needed for a reasonable analysis, limited computer time prevented us from performing such a study.

We also note that the comparison of  $C(\gamma)$  and  $C(\gamma = \emptyset)$  as  $\gamma$  becomes infinitely large could give information on the roughening transition, since an  $M$  with  $\partial M = \gamma$  large may be interpreted as an interface (see Section 2). Comparison of our numerical results with the known value for a phase transition point at  $\beta \sim 0.78$  show, however, that the observables we use here are not suited for such a purpose.

In Section 2 we give the definition of the model. Section 3 contains a presentation and discussion of the numerical results. Finally in Section 4 we give a proof of the existence of the thermodynamic limit of the model.

## 2. DEFINITION OF THE MODEL

In this article the embedding space will be three-dimensional. Note, however, that in the continuum version there is Nash's theorem,<sup>(27)</sup> which states that any compact Riemannian manifold of dimension  $n$  may be isometrically embedded in an  $E^N$  with  $N = (n/2)(3n + 11)$ . In particular for two-dimensional manifolds  $N = 17$ . Hence the choice of  $E^3$  also corresponds to some form of cutoff.

Consider the cubical lattice  $(aZ)^3 \subset E^3$ , i.e., the set of points  $x = (x_1, x_2, x_3)$  in  $E^3$  with  $x_j a^{-1} \in Z (j = 1, 2, 3, a > 0)$ . In the usual way, we obtain  $i$ -cells  $c^i$ . The 0-cells or vertices are the points of the lattice, the open 1-cells are the (nonoriented) bonds. As sets they corresponds to open intervals, whose end points are any two neighboring vertices. The 2-cells are called plaquettes. As sets they are the open squares formed by taking the convex hull of any four neighboring vertices which lie in the same 2-plane. Finally the 3-cells are the open cubes. Discrete versions of  $C(\gamma)$  may now be obtained as follows. First let  $\gamma$  be any finite family of closed 1-cells, such

that every vertex of  $\gamma$  is contained in exactly two closed 1-cells, i.e.,  $\gamma$  forms a union of nonintersecting, piecewise linear Jordan curves in  $E^3$ . In the numerical calculations presented below we will consider two typical cases: In the first case  $\gamma$  is empty and in the second case  $\gamma$  forms the boundary of a square.

Our discrete version  $M$  of a manifold is now what is usually called a two-dimensional cell complex, i.e., a finite family of 2-cells together with the 0-cells and 1-cells contained in their closure. In addition we require that  $\gamma$  is exactly the family of 1-cells in  $M$  which are contained in the closure of an odd number of 2-cells of  $M$ . Let  $C^1(\gamma)$  be the (infinite) family of cell complexes thus obtained. The advantage of considering  $C^1(\gamma)$  will be its close relationship to the three-dimensional Ising model to be discussed below. Note, however, that  $C^1(\gamma)$  contains self-intersecting "manifolds." Therefore we also consider a family of self-avoiding manifolds, given as a subset  $C^2(\gamma)$  of  $C^1(\gamma)$  in the following way.  $M \in C^1(\gamma)$  belongs to  $C^2(\gamma)$  iff any 1-cell is contained in the closure of at most two 2-cells of  $M$  (this condition makes  $M$  a pseudomanifold; see, e.g., Ref. 28). We also want to avoid corner contacts, so we require in addition that no vertex of  $M$  is contained in the closure of six 1-cells of  $M$ . Note that in none of these two models do the manifolds fold onto themselves. More importantly these manifolds are not necessarily connected.

We now put these considerations into an analytic form. We consider any  $i$ -cell  $c^i$  in  $(aZ)^3$  to be a function on  $C^1(\gamma)$  by

$$c^i(M) = \begin{cases} 1 & \text{if } c^i \text{ is in } M \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

In what follows we will identify cells with these functions they define.

The area and the Euler characteristic now become functions on  $C^1(\gamma)$  as follows:

$$\begin{aligned} \text{Area}(M) &= a^2 \sum_{c^2} c^2(M) \\ &= a^2 \#(c^2 \in M) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \chi(M) &= \sum_{c^i} (-1)^i c^i(M) \\ &= \#(c^0 \in M) - \#(c^1 \in M) + \#(c^2 \in M) \end{aligned} \quad (2.3)$$

Note the following a priori estimate

$$|\chi(M)| \leq 12a^{-2} \text{Area}(M) \quad (2.4)$$

which reflects the fact that the lattice formulation acts as a short distance regularization.

We may express  $c^0$  and  $c^1$  and hence  $\chi(M)$  in terms of the basic dynamic  $c^2$  variables as follows. Using the relation  $\text{Max}(x, y) = x + y - xy$ , valid for any  $x$  and  $y$  which only take the values 0 and 1, an easy calculation gives

$$c^1 = \sum'_{1 \leq |I| \leq 4} (-1)^{|I|+1} \prod_{i \in I} c_i^2 \tag{2.5}$$

$c_i^2 \ni c^1$

The prime in  $\sum'$  denotes the restriction that the  $c_i^2$  with  $i$  in  $I$  are pairwise disjoint. Similarly

$$c^0 = \sum'_{1 \leq |I| \leq 8} (-1)^{|I|+1} \prod_{i \in I} c_i^2 \tag{2.6}$$

$c_i^2 \ni c^0$

As a consequence the Euler characteristic is a polynomial of order 8 in the  $c^{2s}$ .

We now turn to a construction of the thermodynamic partition functions. For any finite region  $A \supset \gamma$  in  $R^3$  let  $C_A^j(\gamma)$  consist of those  $M$  in  $C^j(\gamma)$  which are contained in  $A$  such that  $\text{dist}(M, \partial A) \geq 1$  ( $j=1, 2$ ). Obviously  $C_A^j(\gamma)$  is a finite set. We will require  $\partial A$  to be piecewise smooth and to have a tubular neighborhood of radius  $> \sqrt{3} + 2$ .

For any  $\beta \geq 0$  we now define partition functions

$$Z_{A,\beta,j,\gamma} = \sum_{M \in C_A^j(\gamma)} \exp[-\beta \text{Area}(M)], \quad j=1, 2 \tag{2.7}$$

and the corresponding free energy density

$$P_{A,\beta,j,\gamma} = \frac{1}{N(A)} \ln Z_{A,\beta,j,\gamma} \tag{2.8}$$

where  $N(A)$  is the number of 2-cells  $c^2$  in  $A$  (we adopt the sign convention of Ref. 29).

Now for  $j=1$ , the case of self-intersecting manifolds, and all dimensions  $d$ , it is well known that these quantities can be expressed in terms of the  $\mathbb{Z}_2$  lattice gauge theory (see, e.g., Ref. 30, 31). In fact, a short calculation shows that  $Z_{A,\beta,1,\gamma}$  is proportional to the unnormalized expectation value for the Wilson loop  $\gamma$  if the inverse temperature  $\beta_g$  for the Wilson action is related to our  $\beta$  by the duality relation  $\tanh \beta_g = \exp(-\beta)$ . In particular the quotient  $Z_{A,\beta,1,\gamma}/Z_{A,\beta,1,\gamma=\emptyset}$  is just the expectation value for the Wilson loop. Hence our results for  $j=1$  pertain to the  $\mathbb{Z}_2$  model (and are known or could be obtained with this translation code).

Correlation functions are given as

$$\langle \Omega \rangle_{A,\beta,j,\gamma} = Z_{A,\beta,j,\gamma}^{-1} \sum_{M \in C_A^j(\gamma)} \Omega(M) \exp[-\beta \text{Area}(M)] \quad (2.9)$$

where  $\Omega(\cdot)$  is any (bounded) function on  $C^1(\gamma)$ . Of particular interest are the  $\Omega$ 's from the set  $\mathcal{E}$  of all polynomials in the basic random variables  $c^2$ . Their expectations give Green's functions, which are of interest in the theory of string quantization. For example  $\langle c^k \rangle_{A,\beta,j,\gamma}$  is the probability that the  $k$ -cell  $c^k$  is part of an  $M \in C_A^j(\gamma)$ . In Section 4 we will establish the following theorem.

**Theorem 2.1.** For all  $\beta \geq 0$  and  $j = 1, 2$  the thermodynamic limit for the free energy density exists if the limit  $A \rightarrow E^3$  is taken in the sense of van Hove (see, e.g., Ref. 28):

$$\lim_{A \rightarrow E^3} P_{A,\beta,j,\gamma} = P_{\beta,j,\gamma} \quad (2.10)$$

Note that  $N(A)/\text{Vol}(A) \rightarrow 3$  as  $A \rightarrow E^3$ .

We prove a similar statement for the correlation functions.

**Theorem 2.2.** For all  $\beta \geq 0$  and  $j = 1, 2$ , the thermodynamic limit exists for the correlation functions

$$\lim_{A \rightarrow E^3} \langle \Omega \rangle_{A,\beta,j,\gamma} = \langle \Omega \rangle_{\beta,j,\gamma}, \quad \phi \in \mathcal{E} \quad (2.11)$$

where the limit again is taken in the sense of van Hove.

In what follows we will set the lattice spacing  $a$  equal to 1, since the lattice constant  $a$  may be incorporated into the definition of  $\beta$ .

Note, however, that this lattice formulation with planar random surfaces lacks an important feature, believed to be relevant in any theory where one wants to perform the scale limit  $a \rightarrow 0$ . Indeed, there is no classical limit theorem for the action, since areas of smooth manifolds in general may not be approximated in this way. This contrasts with Regge calculus,<sup>(32)</sup> which works with simplexes instead. There one has a classical limit theorem not only for the volume but also for all Lipschitz-Killing curvatures, in particular the scalar curvature and the Euler characteristic.<sup>(33)</sup> In particular the simplicial structures allow one to view the bond lengths as dynamical variables (see, e.g., Refs. 34–37) leading to an ansatz for lattice gravity. Furthermore, one can imagine randomizing the simplicial structure (see, e.g., Refs. 36–38). This could lead to the intriguing context, where also the space-time dimension becomes a



dynamic observable. Returning to our present context, our lattice formulation may therefore not suffice, if one is interested in more detailed structures of  $C(\gamma)$ . As an example we may also mention the recent analysis concerning the Hausdorff dimension of random surfaces with a lattice formulation involving triangles (but with fixed incidence matrix).<sup>(39,40)</sup>

To obtain some insight into our statistical ensembles, it is instructive to note that for the case  $\gamma = \emptyset$ , the ensemble  $C^1(\gamma = \emptyset)$  is in a one-to-two correspondence with the configurations of the Ising model in three dimensions. This is well known (see, e.g., Ref. 41); in fact an element  $M$  in  $C^1(\gamma = \emptyset)$  is nothing but the Peierls interface for a configuration of the Ising model. More precisely, to each 3-cell  $c^3$  associate an Ising variable  $\sigma(c^3) = \pm 1$  and let the action as usual be  $J \cdot \sum \sigma(c^3) \sigma(c^{3'})$ , where the sum is over nearest neighbors, i.e., those pairs  $c^3$  and  $c^{3'}$  whose closures intersect in a 2-cell  $c^2$ . For a given Ising configuration, i.e., given values of the  $\sigma(c^3)$ , we construct  $M \in C^1(\gamma = \emptyset)$  by the condition that  $c^2 \in M$  iff  $\sigma(c^3) \sigma(c^{3'}) = -1$ , where  $c^3$  and  $c^{3'}$  are the two unique 3-cells in the closure of which  $c^2$  is contained. Conversely, to any such  $M$  correspond exactly two Ising model configurations differing by an overall minus sign. Also the actions are equal provided one makes the identification

$$2J = \beta \tag{2.12}$$

Moreover, this correspondence takes the explicit form

$$c^2 = \frac{1}{2}(1 - \sigma(c^3) \sigma(c^{3'})) \tag{2.13}$$

where  $c^2$ ,  $c^3$ , and  $c^{3'}$  are related as above.

Recall also that the Ising model for  $d=3$  is dual to the  $Z_2$  lattice gauge theory (see, e.g., Ref. 31).

A simple calculation now shows that

$$c^1 = 1 - \frac{1}{8} \sum'_{c_1^3 \cap c_2^3 \ni c^1} \sigma(c_1^3) \sigma(c_2^3) - \frac{1}{8} \sigma(c_1^3) \sigma(c_2^3) \sigma(c_3^3) \sigma(c_4^3) \tag{2.14}$$

Here  $c_1^{-3} \cap c_2^{-3} \cap c_3^{-3} \cap c_4^{-3} = c^1$  and again  $\sum'$  denotes that the sum is over pairs of different 3-cells. With the same convention we also have

$$c^0 = 1 - \frac{1}{128} - \frac{1}{128} \sum'_{\substack{2 \leq |I| \leq 8 \\ |I| \text{ even} \\ c_i^3 \ni c^0 (i \in I)}} \prod_{i \in I} \sigma(c_i^3) \tag{2.15}$$

The invariance of (2.13)–(2.15) under the global  $Z_2$  symmetry transformation  $\sigma \rightarrow -\sigma$  reflects the one-to-two correspondence with the Ising model configurations explained above.

This calculation holds for the infinite volume limit case. In a finite volume  $A$  these expressions are still valid for  $c^i$  ( $i=0, 1, 2$ ) in  $\text{int } A$  (the interior of  $A$ ), while there will be modifications at the boundary  $\partial A$ . [Also the boundary conditions may be implemented by the choice  $\sigma(c^3) = 1$  for  $\text{dist}(\partial A, c^3) = 0$ , thus reducing the above one-to-two correspondence to a one-to-one correspondence.] These corrections may be calculated, and below we give them in Section 3 when we compare the analytic solution for  $\beta = 0$  with the numerical results of the computer simulation.

The main point now is that, when  $\gamma = \emptyset$ , the area and the Euler characteristic are certain polynomials in the Ising model variables  $\sigma(c^3)$  of order 2 and 8, respectively. Moreover we have the obvious bounds

$$\begin{aligned} \text{Area} &\leq 3 \text{Vol}(A) \\ |\chi| &\leq 7 \text{Vol}(A) \end{aligned} \tag{2.16}$$

Hence, for those  $\beta$  for which there is a unique thermodynamic limit for the Ising model correlations, the expectation values of the area density  $\text{Area}/\text{Vol}(A)$  and the Euler characteristic density  $\chi/\text{Vol}(A)$  have a finite limit in the ensemble  $C^1(\gamma = \emptyset)$  (similarly Theorem 2.2 for  $j = 1$  is a consequence of corresponding results for the Ising model). Now this property extends to the ensembles  $C^j(\gamma)$  ( $j = 1, 2$ ) in the following way. In Section 4 we will give the proof of the following theorem.

**Theorem 2.3.** For  $j = 1, 2$ , when  $A \rightarrow E^3$  in the sense of van Hove, the limits for the expectations of the area and Euler characteristic volume densities exist in any of the ensembles  $C^j(\gamma)$  in the following cases.

For almost all  $\beta \geq 0$ :

$$\lim_{A \rightarrow E^3} \frac{1}{\text{Vol}(A)} \langle \text{Area} \rangle_{A, \beta, j, \gamma} = \langle \text{Area}_d \rangle_{\beta, j, \gamma} \tag{2.17}$$

For all  $\beta \geq 0$ :

$$\lim_{A \rightarrow E^3} \frac{1}{\text{Vol}(A)} \langle \chi \rangle_{A, \beta, j, \gamma = \emptyset} = \langle \chi_d \rangle_{\beta, j, \gamma = \emptyset} \tag{2.18}$$

The subscript  $d$  has been introduced to denote densities. If  $\gamma = \emptyset$ , the limit in (2.17) exists for all  $\beta \geq 0$ .

This theorem gives the first important consequence: For generic  $\beta \geq 0$  with probability one the area and the Euler characteristic are infinite in the corresponding ensembles  $C^j(\gamma)$ .

The next theorem gives a partial result concerning the comparison of theories with different  $\gamma$ , which we recall for simplicity are assumed to be the boundaries of squares.

**Theorem 2.4.** The difference of free energies in a finite volume  $A$  is uniformly bounded in  $A$ : The estimate

$$|\ln Z_{A,\beta,j,\gamma} - \ln Z_{A,\beta,j,\gamma=\emptyset}| \leq C \text{Area}_{\min}(\gamma) \tag{2.19}$$

holds uniformly in  $A$  with a constant  $C$  depending only on  $\beta$ . A similar estimate holds if the action in addition contains a term proportional to the Euler characteristic.

Although all proofs are carried out for the embedding space  $E^3$ , they may easily be extended to arbitrary  $E^d$ . An alternative existence proof for the thermodynamic limit of (2.10) with an improved estimate in the case  $j=1$  is given in Ref. 42.

As far as macroscopic observables per unit volume are concerned, not surprisingly the ensembles with and without boundary are identical. In fact, by (2.10) the free energies per unit volume are the same. The content of Theorem 2.4 is therefore that one must look at differences of the free energies. They provide the information on the effect of the boundary. We will interpret the difference in (2.10) as the polarization energy of the boundary. If we calculate this polarization energy per unit of the enclosed area, i.e., divide by  $\text{Area}_{\min}(\gamma)$ , then by the correspondence to the  $\mathbb{Z}_2$  lattice gauge theory for  $j=1$ , we may interpret the resulting quantity as the string tension in the limit  $\gamma \rightarrow \infty$ .

Next by Theorem 2.4, the correspondence with the  $\mathbb{Z}_2$  lattice gauge theory for  $j=1$  and provided the thermodynamic limit exists (in the case  $j=2$ ), it makes sense to define the “glueball mass” as the exponential decay rate as  $|x| \rightarrow m$  of the following loop-loop correlation given in terms of polarization energies of boundaries:

$$\lim_{A \rightarrow E^3} [\ln Z_{A,\beta,j,\gamma(x) \cup \gamma} - \ln Z_{A,\beta,j,\gamma=\emptyset} - 2(\ln Z_{A,\beta,j,\gamma} - \ln Z_{A,\beta,j,\gamma=\emptyset})] \tag{2.20}$$

(see, e.g., Ref. 43 for a definition in the same spirit).

The theory of surface roughening may also be incorporated into the present context. This goes as follows. For simplicity let  $A$  be a rectangular box. Take any plane  $P$  which cuts  $A$  into two parts  $A_1$  and  $A_2$  and let now  $\gamma = \partial AP$  be the intersection of  $P$  with the boundary  $\partial A$  of  $A$ . Then  $P$  is the minimal surface for the boundary  $\gamma$  and it is the separating interface when all  $\sigma(e^3)$  are  $+1$  in  $A_1$  and  $-1$  in  $A_2$ , say.

### 3. NUMERICAL RESULTS

In this section we present the numerical results obtained from a Monte Carlo computer simulation. Since our aim is to exhibit qualitative features of our model, which we consider interesting in their own right, apart from thermalization and normalization checks no attempt has been made to obtain precise numerical data and to estimate errors. We measured both in the ensemble  $C^1(\gamma)$  (self-intersecting manifolds) and in the ensemble  $C^2(\gamma)$  (self-avoiding manifolds). The Monte Carlo method we used is a local heat bath motivated by the close relationship to the three-dimensional Ising model as explained in Section 2. It has already been used by Sterling and Greensite.<sup>(41)</sup> There is an alternative local heat bath method,<sup>(44)</sup> which incorporates the self-avoiding condition and which was inspired by a new proposal to evaluate Fermion determinants in lattice field theories.<sup>(45)</sup> For yet other computer simulations of random surfaces, see, e.g., Refs. 39, 46–49.

The upgrading procedure we use may now be described as follows. At each instant the computer has stored a manifold  $M \in C^j(\gamma)$  by assigning the values 0 or 1 to each 2-cell  $c^2$  in  $A$ , where the 2-cells whose closure intersects the boundary  $\partial A$  are being given the constant value 0. The computer now sweeps through all 3-cells  $c^3$  away from  $\partial A$ . At each  $c^3$  a new trial manifold  $M_{\text{new}}$  out of the present manifold  $M_{\text{old}}$  is constructed by performing a local surgery in the form of the substitution

$$c_{\text{old}}^2 \rightarrow c_{\text{new}}^2 = c_{\text{old}}^2 + 1 \pmod{2} \quad (3.1)$$

for all  $c^2$  lying in the closure of the given  $c^3$ . It is easy to see that  $M_{\text{old}} \in C^1(\gamma)$  implies  $M_{\text{new}} \in C^1(\gamma)$ . Additional checks have to be made if instead one works with  $C^2(\gamma)$  to ensure that  $M_{\text{new}}$  lies in this ensemble (and if not, one proceeds to the next 3-cell). As usual for such a method, in each case the new manifold is then accepted with probability

$$w = \exp\{-\beta \Delta \text{Area} / [1 + \exp(-\beta \Delta \text{Area})]\} \quad (3.2)$$

by calling up a random number in the interval  $[0, 1]$ . Here  $\Delta \text{Area} = \text{Area}(M_{\text{new}}) - \text{Area}(M_{\text{old}})$  is a local quantity, which depends only on the values at the 2-cells contained in the closure of the given 3-cell. In the next step the same procedure is repeated at the next 3-cell and so on. This upgrading procedure satisfies detailed balance. In fact, for the ensemble  $C^1(\gamma)$  this is a standard argument, and for the ensemble  $C^2(\gamma)$  this is most easily seen by imagining excluded configurations to have infinite energy. In Ref. 39 we show for  $C_A^1(\gamma)$  that the Markov process defined by the upgrading is ergodic, thus making the Gibbs' distribution the unique

equilibrium distribution. Recall that in situations like these ergodicity is ensured, if the set of configurations is connected under the Markov process.

We performed calculations on a  $A = 18 \times 18 \times 10$  lattice and measured the case when  $\gamma$  was empty or the boundary of a square in the 1–2-direction placed in the middle. We started at  $\beta_0 = 1.8$  for self-intersecting manifolds and at  $\beta = 1$  for self-avoiding manifolds. The initial configuration was the empty manifold for  $\gamma = \emptyset$  and the minimal surface for  $\gamma \neq \emptyset$ . The system was heated gradually with  $\Delta\beta = 0.05$ , using the final manifold for  $\beta_i$  as the initial manifold for  $\beta_{i+1}$  and going down to  $\beta = 0$ . In the case of self-intersecting manifolds the number of sweeps for given  $\beta_i$  was 150, of which the last 100 were used to evaluate averages. For self-avoiding manifolds the corresponding numbers were 300 and 200. The upgrading time per 3-cell for self-avoiding manifolds was on the average about 3 times as large as for self-intersecting manifolds, where no restraints had to be checked.

As a check of thermalization for self-intersecting manifolds, we used the fact that at  $\beta = 0$  and for  $\gamma = \emptyset$  the relation to the Ising model allows one to evaluate expectations explicitly. In particular in the infinite volume limit the relations (2.13)–(2.15) give

$$\begin{aligned} \langle \text{Area}_d \rangle_{\beta=0, j=1, \gamma=\emptyset} &= 3/2 = 1.5 \\ \langle \chi_d \rangle_{\beta=0, j=1, \gamma=\emptyset} &= (1 - 1/128) - 3(1 - 1/8) + 3/2 = -0.617 \end{aligned} \tag{3.3}$$

for the area density and the density of the Euler characteristic. More generally, at  $\beta = 0$ , finite size effects may be calculated. Let  $A$  be a box with sides  $l_1, l_2$ , and  $l_3$ . Since any manifold  $M$  lives in the interior of  $A$  such that  $\text{dist}(\partial A, M) \geq 1$ , we take as the volume the quantity  $\text{Vol} = (l_1 - 2)(l_2 - 2)(l_3 - 2)$ . An easy calculation then gives

$$\begin{aligned} \langle \text{Area} \rangle_{A, \beta, j=1, \gamma=\emptyset} / \text{Vol} \\ = 3/2 + (1/2)[1/(l_1 - 2) + 1/(l_2 - 2) + 1/(l_3 - 2)] \end{aligned} \tag{3.4}$$

Using (2.13)–(2.15) an analogous but somewhat lengthy formula may be derived for the Euler characteristic.

As a check of thermalization for self-avoiding manifolds we started with different seeds in the random number generator. Also we compared with the result obtained by averaging over the last 100 sweeps only.

Figure 1 gives a plot of the area density versus  $\beta$  for self-intersecting manifolds and for the case  $\gamma = \emptyset$ , which by our previous discussion is essentially the energy density expectation of the three-dimensional Ising model. By (3.4) it should take the value 1.625 at  $\beta = 0$ . Numerically this is well reproduced. Note also that by Schwarz inequality the derivative with respect to  $\beta$  is negative, so the area density is monotonically decreasing in

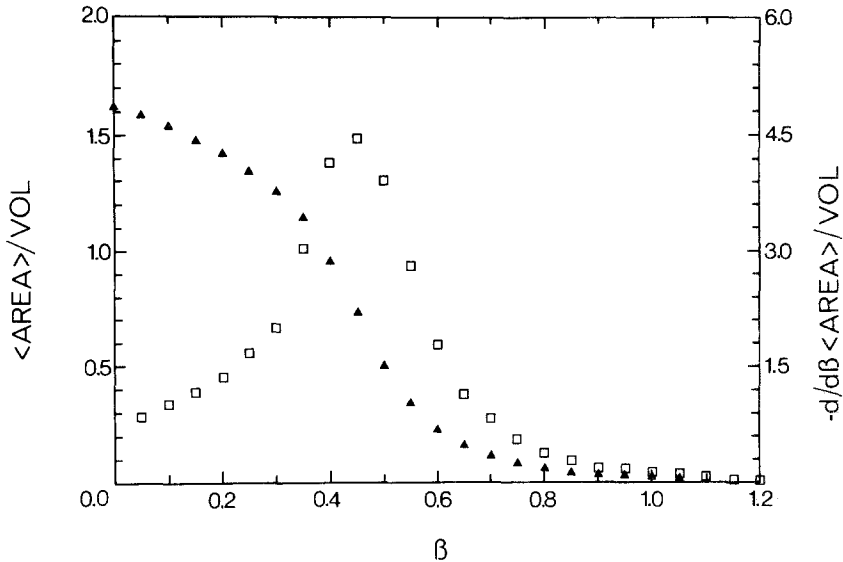


Fig. 1. The expectation of the area density ( $\blacktriangle$ ) and (minus) its derivative ( $\square$ ) versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-intersecting manifolds without boundary.

$\beta$ . Figure 1 also contains a plot of (minus) this derivative, reproducing (with our accuracy) a phase transition point of the three-dimensional Ising model at the right  $\beta$ , namely, at  $\beta = 0.4338$  [see (2.12) and, e.g., Ref. 50]. The corresponding result for the self-avoiding manifold is similar with a corresponding phase transition at  $\beta \sim 0.3$ . Also the area density is always smaller than for self-intersecting manifolds, as is to be expected.

Figure 2 shows the Euler characteristic per unit area plotted versus  $\beta$  and again for self-intersecting manifolds but now with boundary  $\gamma = \partial(8 \times 8)$ . By this observable we mean the following. Consider the quantity

$$G(M) = \chi(M)/\text{Area}(M) \quad (3.5)$$

which for  $\gamma = \emptyset$  can be viewed as the mean Gauss–Bonnet density of the manifold  $M$ . By the bound (2.4), this is an intensive quantity. For large  $\beta$ , intuitively  $M$  is approximately the minimal surface  $M_{\min}(\gamma)$  plus small and not densely populated connected components with empty boundary. For decreasing  $\beta$ ,  $M$  starts to fill up all of  $A$  by forming more and more connected components per unit area (or equivalently per unit volume). At a certain  $\beta$ ,  $\langle G \rangle$  attains a maximum, indicating that the increase in the number of handles per unit area is greater than the increase in the number of connected components per unit area. Eventually, as  $\beta \rightarrow 0$ ,  $\langle G \rangle$  becomes negative. More precise measurements<sup>(44)</sup> indicate that the temperature, at

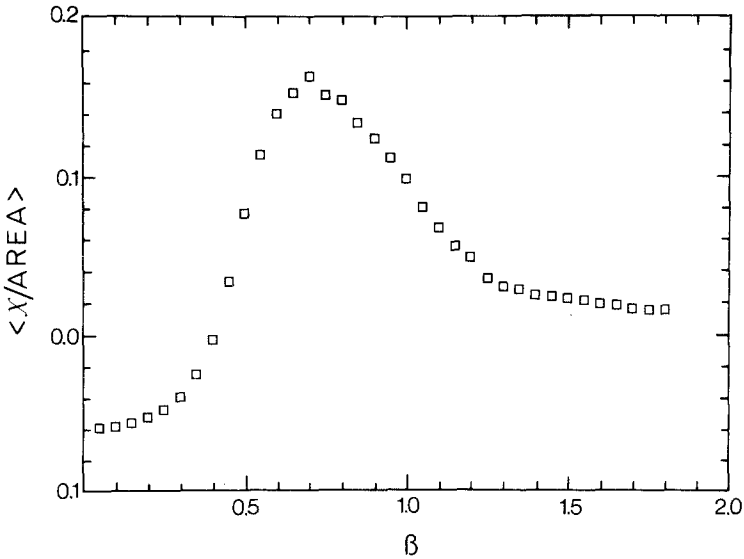


Fig. 2. The expectation of the Euler characteristic per unit area versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-intersecting manifolds with boundary  $\gamma = \partial(8 \times 8)$ .

which  $\langle G \rangle = 0$ , is exactly the critical temperature of the Ising model. This might lead to a new geometrical understanding of the structure of the phase transition in the Ising model. Also we obtained a qualitatively similar picture for the case of self-avoiding manifolds.

In Fig. 3 for  $\gamma = \partial(8 \times 8)$ ,  $\partial(12 \times 12)$ ,  $\partial(18 \times 18)$ , we have plotted the following dimensionless quantity versus  $\beta$ :

$$(\langle \text{Area} \rangle_{A,\beta,j=1,\gamma} - \langle \text{Area} \rangle_{A,\beta,j=1,\gamma=\emptyset}) / \text{Area}_{\min}(\gamma) \tag{3.6}$$

the area excess, and as discussed above it is obtained by taking the derivative of the polarization energy of the boundary. By our previous discussion, for  $\gamma \rightarrow \infty$  this may be interpreted as the derivative of the string tension with respect to  $\beta$ . Figure 4 gives another dimensionless quantity

$$\langle \chi \rangle_{A,\beta,j=1,\gamma} - \langle \chi \rangle_{A,\beta,j=1,\gamma=\emptyset} \tag{3.7}$$

The quantity in (3.7) is the excess of the Euler characteristic. By Theorem 2.4 and the discussion thereafter we expect that they are intensive quantities. Since they are differences of extensive quantities, their numerical evaluation is subject to strong fluctuations. Thus the interesting theoretic aspects of this model seem simultaneously difficult to analyze numerically. For the polarization energy in the ensemble  $C^1(\gamma)$ , however, we show in

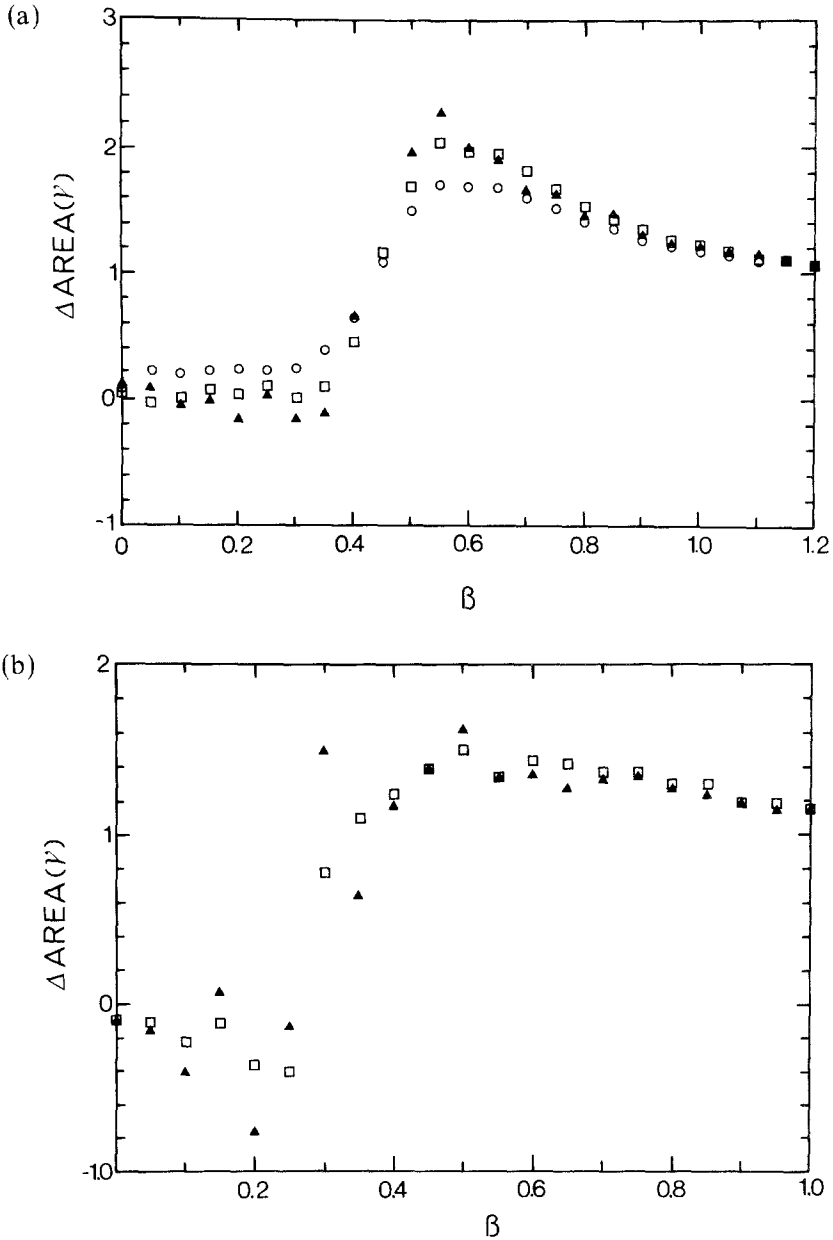


Fig. 3. (a) The relative area excess (3.6) versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-intersecting manifolds with boundary  $\gamma = \partial(8 \times 8)$  ( $\blacktriangle$ ),  $\gamma = \partial(12 \times 12)$  ( $\square$ ) or  $\gamma = \partial(18 \times 18)$  ( $\circ$ ). (b) The relative area excess (3.6) versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-avoiding manifolds with boundary  $\gamma = \partial(8 \times 8)$  ( $\blacktriangle$ ),  $\gamma = \partial(12 \times 12)$  ( $\square$ ).



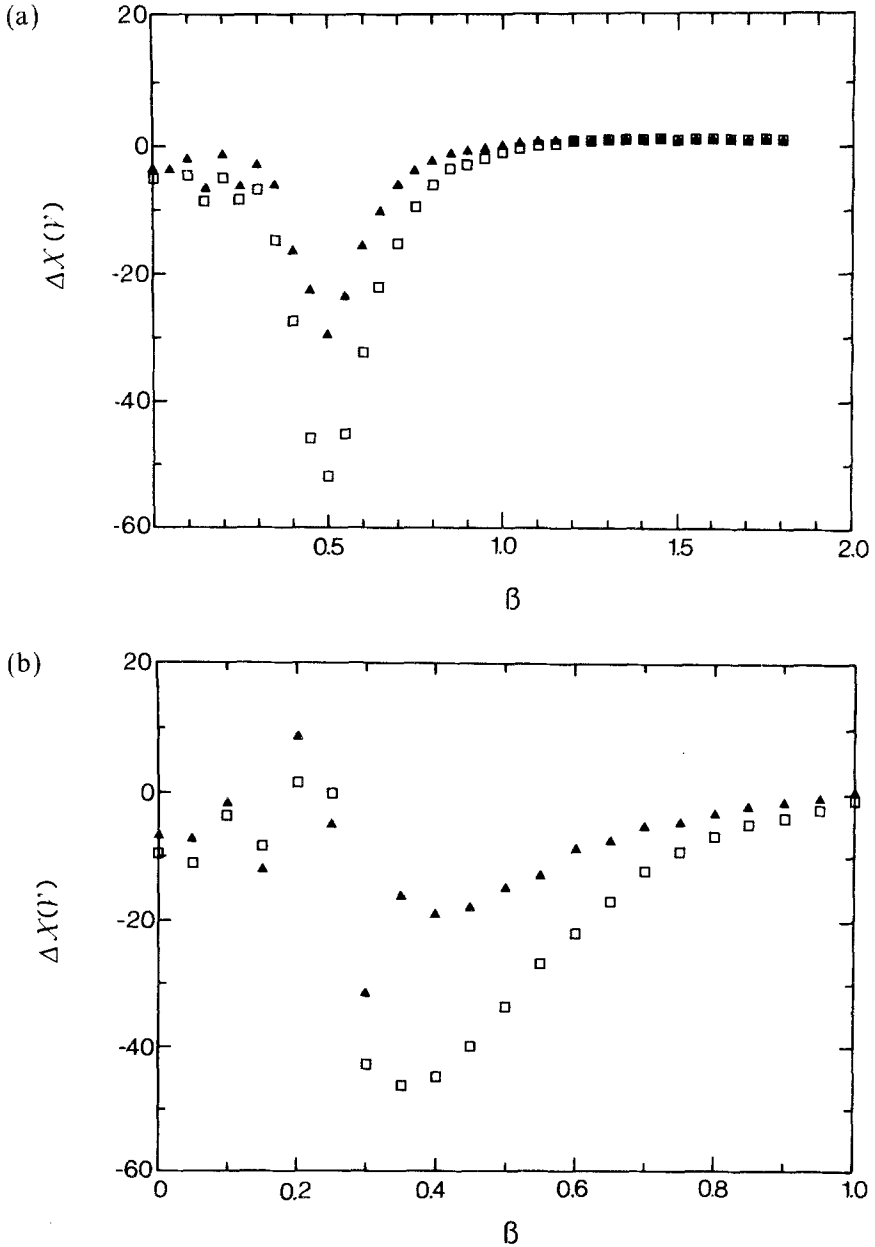


Fig. 4. (a) The Euler characteristic excess (3.7) versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-intersecting manifolds with boundary  $\gamma = \partial(8 \times 8)$  ( $\blacktriangle$ ),  $\gamma = \partial(12 \times 12)$  ( $\square$ ). (b) The Euler characteristic excess (3.7) versus  $\beta$  on a  $18 \times 18 \times 10$  lattice for self-avoiding manifolds with boundary  $\gamma = \partial(8 \times 8)$  ( $\blacktriangle$ ) or  $\gamma = \partial(12 \times 12)$  ( $\square$ ).

Ref. 42 that it may be obtained from expectations in the ensemble  $C^1(\gamma = \emptyset)$  alone, thus avoiding this difficulty. In Ref. 42 we also establish the relationship of the polarization energy to the string tension.

At small  $\beta$  the manifolds intuitively should lose memory of their boundary, hence we expect the quantities (3.6) and (3.7) then to take the value 0. In particular the string tension should vanish. Numerically this is reasonably well satisfied. Furthermore both plots indicate a phase transition at  $\beta \sim 0.5$  and at  $\beta \sim 0.3$  for self-intersecting and self-avoiding manifolds, respectively. We interpret these as the  $\beta$ -values where this memory of the boundary is lost, i.e., where the long-range correlations disappear. These phase transitions are scale invariant in the sense of a  $\gamma$  independence. We recall that a corresponding observation has been made for the self-avoiding random walk. There the phase transition point does not depend on the choice of the (fixed) end points.<sup>(51,52)</sup> Considerations in this spirit go back to Hammersley.<sup>(53)</sup> It would therefore be interesting to see how much our critical value for  $\beta$  will differ from the one in a lattice model in which the size of  $\gamma$  is allowed to vary. On a computer such an ensemble could for example be simulated by adapting the method used by A. Beretti and A. Sokal for random walks with varying end points.<sup>(54)</sup> Note also that for the case of self-avoiding manifolds the choice  $\partial(8 \times 8)$  for the boundary seems to be too small, since the manifolds are then not yet "elastic" enough.

Next notice that the choice  $\gamma = \partial(18 \times 18)$  corresponds to the interface situation. Now the roughening phase transition for the Ising model is located at  $\beta = 0.78$  (see, e.g., Refs. 55, 56, and the references quoted there). Not surprisingly our observables plotted in Figs. 3a and 4a do not exhibit a singularity there. Indeed, the roughening transition is obtained by considering transverse fluctuations of the interface. Note, however, that the maximum of the mean Gauss-Bonnet density, as plotted in Fig. 2, is located close to this  $\beta$ -value. (For other discussions of the roughening transition in the context of Wilson loops, see, e.g., Refs. 57-59).

The plot in Fig. 5 describes the result of a test of the statistical independence of the (time) fluctuations  $\delta \text{Eul}(t)$  and  $\delta \text{Area}(t)$  of the Euler characteristic and the area as discussed in Section 1. This test was carried out for self-intersecting manifolds with a new simulation on a  $22 \times 22 \times 22$  lattice with empty boundary starting at  $\beta = 0.6$  going down to  $\beta = 0.0$  in steps  $\Delta\beta = 0.02$ . The number of sweeps was again 150, of which the last 100 were used to calculate averages.

Plotted is the quantity

$$\left[ \frac{\delta \text{Eul}(t)}{\delta \text{Area}(t)} \right]^2 \cdot \frac{\overline{\delta \text{Area}(t)^2}}{\overline{\delta \text{Eul}(t)^2}} \quad (3.8)$$

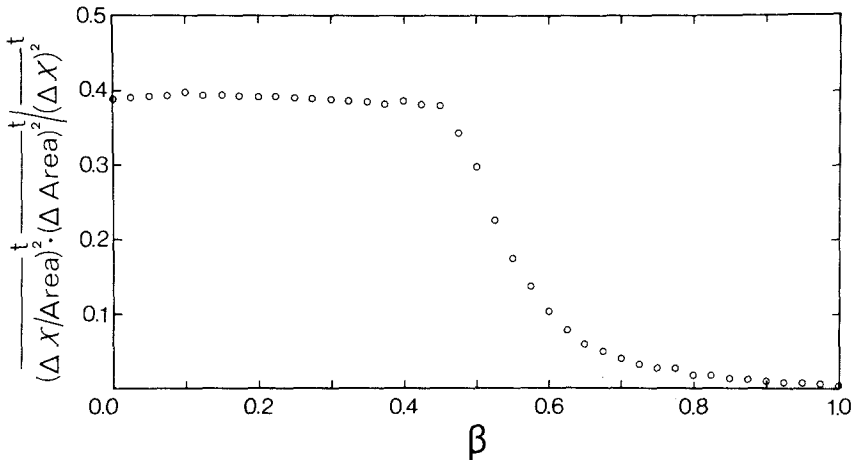


Fig. 5. Test of statistical independence of time fluctuations of area and Euler characteristic by measuring quantity (3.8) versus  $\beta$  on a  $22 \times 22 \times 22$  lattice for empty boundary.

The results are quite satisfactory. For large  $\beta$  these quantities are small compared to 1, indicating that for such  $\beta$  changes in topology are strongly correlated to changes in the areas. Close to the critical point there is a sharp increase and finally a constant behavior for small  $\beta$ . Note that since we compare the manifolds  $M(t)$  and  $M(t + 1)$ , these time averages do not correspond to averages in the Gibbs ensemble.

#### 4. THERMODYNAMIC LIMITS

This section contains a general proof of thermodynamic limits for Gibbs' distributions for a large class of potentials on  $C^j(\gamma)$  in the spirit of classical lattice systems as discussed, e.g., in Ref. 29.

Theorems 2.1–2.3 will therefore be particular cases of the results to be established below. Compared to the typical situations considered in Ref. 29, the additional difficulty in proving thermodynamic limit theorems in the present context stems from the geometric conditions imposed on the  $M \in C^j(\gamma)$ . These boundary (and self-avoiding) properties may be viewed as being described in terms of hard core potentials. The main idea of the proof will be to prove approximate sub- and superadditivity of the free energy densities by performing local surgeries on the manifolds  $M \in C^j(\gamma)$ . Note that techniques of the same spirit have been used in percolation theory.<sup>(60,61)</sup>

We start by considering the following class of real-valued functions on  $C^j_A(\gamma)$ . Let  $U_A \in \mathcal{E}$  be any polynomial of the  $c^{2s}$  contained in  $A$ . If we

enumerate these  $c^{2s}$ s and  $N(A)$  is their number, for given values of the  $c^{2s}$ s we may write

$$U_A(c_1^2, \dots, c_{N(A)}^2) = U_A(X) \quad (4.1)$$

where  $X = \{c^2 \in A, c^2 = 1\}$ . With this notation we assume that

$$U(X) = \sum_{X \subset Y} \phi(Y) \quad (4.2)$$

where the "potential"  $\phi$  is a function on finite subsets of any of our  $c^{2s}$ s in  $R^3$ . This assumption in particular implies that  $U_A(X)$  does not depend on the index  $A$ , which therefore we will drop from now on.

The translation group on  $Z^3$  induces translations on the plaquettes and hence on these subsets  $X$  of  $c^{2s}$ s. Let  $B$  be the real Banach space of functions  $\phi$  which are translation invariant

$$\phi(X + a) = \phi(X) \quad (4.3)$$

for all  $a \in Z^3$  and for which

$$\|\phi\| = \sum' \frac{|\phi(X)|}{N(X)} < \infty \quad (4.4)$$

Here the prime denotes the restriction to those  $X$  for which the origin 0 lies in the closure of at least one  $c^2 \in X$ . Also  $N(X)$  is the number of  $c^{2s}$ s in  $X$ , i.e., the "area" of  $X$ .

The area function itself belongs to this class and is given by  $\phi(X) = 1$  iff  $X$  consists of exactly one  $c^2$  and is zero otherwise. Its norm is 6. Analogously, as may be inferred from (2.13)–(2.15), the Euler characteristic also belongs to this class. Next let  $B_0$  be the dense linear subspace consisting of those finite range potentials  $\phi$  with  $\phi(X) \neq 0$  for only a finite number of  $X$  having the origin 0 in their closure. Now define

$$U_\phi(X) = \sum_{Y \subset X} \phi(Y) \quad (4.5)$$

Then if  $\phi \in B$

$$|U_\phi(X)| \leq N(X) \|\phi\| \quad (4.6)$$

Let  $Q$  be any polynomial in the  $c^{2s}$ s and consider  $Q$  to be a function of  $M$ .  $Q$  is introduced to obtain correlation functions. By  $\text{supp } Q$  we denote the smallest box containing all the  $c^{2s}$ s in  $Q$ .

A Gibbs' ensemble on  $C_A^j(\gamma)$  is now given by attributing to each "manifold"  $M$  a weight

$$\exp[-U_\phi(M) - Q(M)] \quad (4.7)$$

We now introduce the partition function

$$Z_{A,j,\gamma}(\phi, Q) = \sum_{M \in C'_A(\gamma)} \exp[-U_\phi(M) - Q(M)] \tag{4.8}$$

and define the free energy density to be

$$Z_{A,j,\gamma}(\phi, Q) = N(A)^{-1} \log Z_{A,j,\gamma}(\phi, Q) \tag{4.9}$$

where we recall that  $N(A)$  is the number of  $c^{2s}$ 's in  $A$ . This generalizes our definition of the partition function and free energy density as given in Section 2.

**Theorem 4.1.** If  $\phi \in B$  then the following limit exists if  $A$  tends to infinity in the sense of van Hove:

$$P_{j,\gamma}(\phi, Q) = \lim_{A \rightarrow E^3} P_{A,j,\gamma}(\phi, Q) \tag{4.10}$$

Furthermore

(a) if  $\phi, \psi \in B$  then

$$|P_{j,\gamma}(\phi, Q) - P_{j,\gamma}(\psi, Q)| \leq \|\phi - \psi\| \tag{4.11}$$

(b) the functions  $P_j(\cdot, Q)$  are convex and positive on  $B$ .

*Proof.* The elementary estimate

$$\left| \log \sum_i \exp(-x_i) - \log \sum_i \exp(-y_i) \right| \leq \text{Max}_i |x_i - y_i| \tag{4.12}$$

for real  $x_i$  and  $y_i$  combined with (4.8)–(4.9) gives

$$|P_{A,j,\gamma}(\phi, Q) - P_{A,j,\gamma}(\psi, Q)| \leq \|\phi - \psi\| \tag{4.13}$$

uniformly in  $A$ .

Similarly the convexity of  $P_{A,j,\gamma}(\phi, Q)$  in  $\phi$  follows from the elementary inequality

$$\begin{aligned} & \log \sum_i \exp(-\alpha x_i - (1 - \alpha) y_i) \\ & \leq a \log \sum_i \exp(-x_i) + (1 - a) \log \sum_i \exp(-y_i) \end{aligned} \tag{4.14}$$

valid for  $0 \leq a \leq 1$  and all real  $x_i$  and  $y_i$ . Indeed (4.14) is an easy consequence of Hölder's inequality. Hence (a) and (b) of Theorem 4.1 will follow, once (4.10) is established. Also by (4.13) and the density of  $B_0$  in  $B$

it suffices to verify (4.10) for  $\phi \in B_0$ . For such a finite range potential  $\phi$ , let  $A$  be a finite union of cubes forming a rectangular box such that  $\phi(X) = 0$  whenever the origin is in the closure of  $X$  and  $X$  is not contained in  $A$ .

Assume now that each connected component of  $\partial A_1 \cap A_2$  or  $A_1 \cap \partial A_2$  is diffeomorphic to the open unit 2-disk and that  $\text{dist}(\text{supp } Q, \partial A_k) > 2$ ,  $\text{dist}(\gamma, \partial A_k) > 2$  ( $k = 1, 2$ ). In particular  $\gamma$  and  $\text{supp } Q$  lie either completely inside or outside each  $A_k$  and  $A_1 \cup A_2$ , so  $\gamma \cap A_k$  is either  $\gamma$  or the empty set ( $k = 1, 2$ ) and similarly for  $\text{supp } Q$ .

Let  $N_1(A_1, A_2)$  be 12 times the number of  $a$ 's for which  $A_1 \cap (A + a) \neq \emptyset$  and  $A_2 \cap (A + a) \neq \emptyset$ . Also let  $N_2(A_1, A_2)$  be the number of  $c^{2s}$ 's with distance smaller than  $\sqrt{3} + 2$  to  $(\partial A_1 \cap A_2) \cup (A_1 \cap \partial A_2)$ . Set

$$N(A_1, A_2) = \text{Max}_j N_j(A_1, A_2) \tag{4.15}$$

By an easy adaption of the arguments in Ref. 29 (see, e.g., the proof of Proposition 2.3.2 there), (4.10) is now a consequence of the following lemma.

**Lemma 4.2.** With the above notation

$$\begin{aligned} & |N(A_1 \cup A_2) P_{A_1 \cup A_2, j, \gamma}(\phi, Q) - N(A_1) P_{A_1, j, \gamma}(\phi, Q) - N(A_2) P_{A_2, j, \gamma}(\phi, Q)| \\ & \leq N(A_1, A_2)(3 \|\phi\| + 2 \ln 2) \end{aligned} \tag{4.16}$$

*Proof.* We establish approximate sub- and superadditivity properties. As for the subadditivity we have

$$\begin{aligned} Z_{A_1 \cup A_2, j, \gamma}(\phi, Q) &= \sum_{\substack{X_1 \subset A_1, X_2 \subset A_2 \\ X_1 \cup X_2 \in C_{A_1 \cup A_2}^j(\gamma)}} \exp[-U_\phi(X_1 + X_2) - Q(X_1 + X_2)] \\ &\leq \sum_{X_1 \subset A_1 \cap C_{A_1 \cup A_2}^j(\gamma)} \exp[-U_\phi(X_1) - Q(X_1)] \\ &\quad \times \sum_{X_2 \subset A_2 \cap C_{A_1 \cup A_2}^j(\gamma)} \exp[-U_\phi(X_2) - Q(X_2)] \\ &\quad \times \exp N(A_1, A_2) \|\phi\| \end{aligned} \tag{4.17}$$

Here we used the estimate

$$|U_\phi(X_1 \cup X_2) - U_\phi(X_1) - U_\phi(X_2)| \leq N(A_1, A_2) \|\phi\| \tag{4.18}$$

which follows by the same arguments as in Ref. 29. Also we used the notation  $X_k \subset A_k \cap C_{A_1 \cup A_2}^j(\gamma)$  to denote the restriction of an element  $M$  in  $C_{A_1 \cup A_2}^j(\gamma)$  to  $A_k$ . We also used the fact that  $Q(X_1 + X_2) = Q(X_1) + Q(X_2)$ , which follows from our assumptions on  $A_1, A_2$  and  $\text{supp } Q$ .

Now  $X_k$  is not necessarily an element of  $C_{A_k}^j(\gamma)$  ( $k=1, 2$ ). However, the failure to satisfy this condition arises only near  $\partial A_1 \cap A_2$  for  $k=1$  and near  $A_1 \cap \partial A_2$  for  $k=2$ . We will therefore modify  $X_1$  near  $\partial A_1 \cap A_2$  to obtain an element  $M_1(X_1) \in C_{A_1}^j(\gamma)$ . This will give

$$\begin{aligned} |U_\phi(X_1) - U_\phi(M_1(X_1))| &\leq N(A_1, A_2) \|\phi\| \\ Q(M_1(X_1)) &= Q(X_1) \end{aligned} \tag{4.19}$$

We will also see that for any given  $X_1$

$$\#\{Y_1 \mid M_1(Y_1) = M_1(X_1)\} \leq 2^{N(A_1, A_2)} \tag{4.20}$$

There is a similar construction of  $M_2(X_2)$ . Replacing the sum over  $X_1$  and  $X_2$  by  $M_1$  and  $M_2$ , respectively, (4.17)–(4.20) then give

$$\begin{aligned} N(A_1) P_{A_1, j, \gamma}(\phi, Q) + N(A_2) P_{A_2, j, \gamma}(\phi, Q) \\ \leq P_{A_1 \cup A_2, j, \gamma}(\phi, Q) + N(A_1, A_2)(3 \|\phi\| + 2 \ln 2) \end{aligned} \tag{4.21}$$

To obtain  $M_1(X_1)$  we proceed as follows. For any component  $T$  of  $\partial A_1 \cap A_2$ , let the two-dimensional manifold  $S$  be constructed in the following way. Consider that part of the boundary of the tubular neighborhood of  $\partial A_1 \cap A_2$  of radius  $\sqrt{3} + 1/2$  which is contained in  $A_1$ . By moving the points of this manifold slightly and differentiably by at most  $1/4$  we may achieve that the resulting set  $S$  does not contain any vertices  $c^0$  and that it intersects any  $c^i$  ( $i=1, 2$ ) transversally (see Fig. 6).  $S$  has the topology of

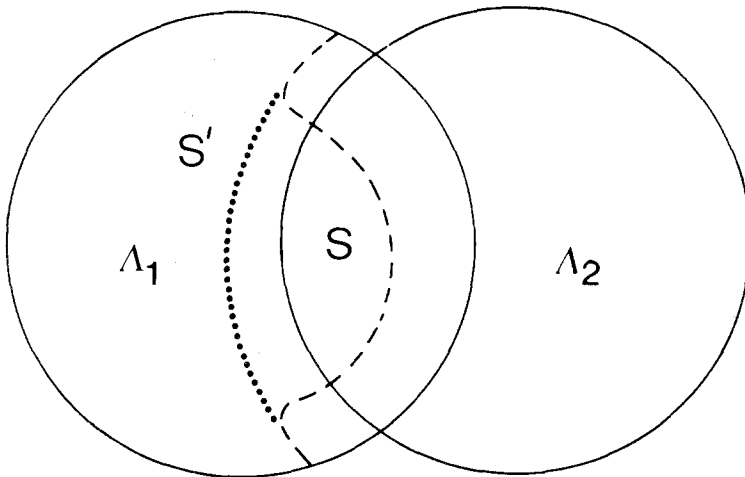


Fig. 6. Surgery near the intersection of two sets  $A_1, A_2$ .

the open unit 2-disk and its intersection with  $X_1$  and hence  $M$  has the typical form depicted in Fig. 7a for the case

$$M \in C^1_{A_1 \cup A_2}(\gamma)$$

and Fig. 7b for the case

$$M \in C^2_{A_1 \cup A_2}(\gamma)$$

We note that by assumption and construction  $S$  does not intersect  $\gamma$  and  $\text{supp } Q$ . Next we assign values 0 and 1 to the regions on  $S$  carved out by the piecewise differentiable curves forming  $S \cap X_1$ : To the region having  $\partial S$  as part of its boundary we give the value 0. Inductively we change the value any time we cross a line in  $S \cap X_1$ . It is easy to see that this procedure is well defined and thus uniquely gives values 0 and 1 to all  $c^3$  cells that intersect  $S$ . We now modify  $X_1$  between  $S$  and the component  $T$  of  $\partial A_1 \cap A_2$  by taking the surface formed by those  $c^3$  that have value 1. Doing this for all components  $T$  leads to the desired manifold  $M_1(X_1)$ . Obviously the estimates (4.19) and (4.20) hold. The proof of (4.19) is analogous to the proof of (4.18) and (4.20) follows by counting at how many  $c^2$ s at most  $X_1$  and  $M_1(X_1)$  may differ. This concludes the proof of the approximate subadditivity.

As for the approximate superadditivity, we start with the relation

$$\begin{aligned} Z_{A_1, j, \gamma}(\phi, Q) \cdot Z_{A_2, j, \gamma}(\phi, Q) = & \sum_{M_1 \in C^j_{A_1}(\gamma)} \sum_{M_2 \in C^j_{A_2}(\gamma)} \exp[-U_\phi(M_1) \\ & - U_\phi(M_2) - Q(M_1) - Q(M_2)] \end{aligned} \quad (4.22)$$

Now it is easy to construct examples where  $M_1 \cup M_2$  is not an element of  $C^j_{A_1 \cup A_2}(\gamma)$ . Again, however, it is possible to perform a surgery to obtain

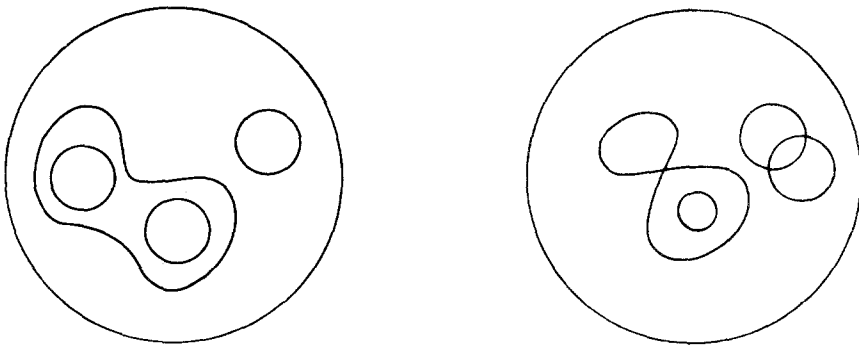


Fig. 7. Curves resulting from the intersection of a manifold in  $C^1(\gamma)$  (a) or in  $C^2(\gamma)$  (b) with the set  $S$ .



$M'_1(M_1) \in C_{A_1}^j(\gamma)$  lying entirely in  $A_1 \setminus A_2$ . This time, for example,  $S'$  in Fig. 6 is used to modify  $M_1$  between  $S'$  and the corresponding component of  $\partial A_2 \cap A_1$  in a completely analogous way as was done above with  $S$ . Applying this method to every component of  $\partial A_2 \cap A_1$  gives an  $M'_1(M_1)$  with

$$|U_\phi(M'_1(M_1)) - U_\phi(M_1)| \leq N(A_1, A_2) \|\phi\| \tag{4.23}$$

In analogy to (4.20) for given  $M_1$ , we have the estimate

$$\# \{ \hat{M}_1 \mid M'_1(\hat{M}_1) = M'_1(M_1) \} \leq 2^{N(A_1, A_2)} \tag{4.24}$$

A similar construction may be carried out for  $M_2$ , giving an  $M'_2(M_2)$ . But then, since by construction

$$M'_1(M_1) \cap M'_2(M_2) = \emptyset \tag{4.25}$$

$M'_1(M_1) \cup M'_2(M_2)$  indeed belongs to  $C_{A_1 \cup A_2}^j(\gamma)$ . Also

$$Q(M_1) + Q(M_2) = Q(M'_1(M_1) \cup M'_2(M_2)) \tag{4.26}$$

Estimates (4.22)–(4.24) combined give

$$\begin{aligned} Z_{A_1, j, \gamma}(\phi, Q) \cdot Z_{A_2, j, \gamma}(\phi, Q) &\leq \sum_{M'_1 \cup M'_2 \in C_{A_1 \cup A_2}^j(\gamma)} \exp[-U_\phi(M'_1) \\ &\quad - U_\phi(M'_2) - Q(M'_1 \cup M'_2)] \\ &\quad \times \exp[2N(A_1, A_2)(\|\phi\| + 2 \ln 2)] \\ &\leq \sum_{M \in C_{A_1 \cup A_2}^j(\gamma)} \exp[-U_\phi(M) - Q(M)] \\ &\quad \times \exp[N(A_1, A_2)(3 \|\phi\| + 2 \ln 2)] \end{aligned} \tag{4.27}$$

Taking the logarithm proves the desired approximate superadditivity, concluding the proof of Lemma 4.2 and hence also of Theorem 4.1. ■

Theorem 2.2 now follows from the simple trick. Let  $Q$  be any monomial in the  $c^{2s}$ s with coefficient 1. Then  $Q$  takes only values 0 and 1 and hence  $Q = (e^{\gamma Q} - 1)(e^\gamma - 1)^{-1}$ . Therefore the expectation of  $Q$  is a sum of quotients of partition functions such that we may apply Theorem 4.1. The first part of Theorem 2.3 follows from a standard convexity argument. As for the second part we note that owing to the assumption  $\gamma = \emptyset$  we have translation invariance. Hence we may apply Theorem 2.2, since  $X$  is a polynomial in the  $c^{2s}$ s.

Finally the proof of Theorem 2.4 is obtained by using the same ideas as employed in the proof of Lemma 4.2. More generally we have the following result. For a given  $\gamma$  forming the boundary of a square, choose a finite set of 3-cells such that their closure forms a cube  $T(\gamma)$  with  $\gamma$  in the interior and  $10 < \text{dist}(x, \gamma) < 12$  for any  $x \in \partial T(\gamma)$ . For a finite range potential  $\phi \in B_0$  and the associated range  $\Delta$ , consider the set  $T(\gamma, \Delta) = T(\gamma) + \Delta$ . Let  $N(\gamma, \Delta)$  be the number of 2-cells intersecting  $T(\gamma, \Delta)$ . Theorem 2.4 is now a special case of the following lemma.

**Lemma 4.3.** The following estimate holds uniformly in  $\mathcal{A}$ :

$$\exp[-N(\gamma, \Delta)(\|\phi\| + \ln 2)] \leq \frac{Z_{\mathcal{A}, j, \gamma}(\phi, Q)}{Z_{\mathcal{A}, j, \gamma = \emptyset}(\phi, Q)} \leq \exp[N(\gamma, \Delta)(\|\phi\| + \ln 2)] \quad (4.28)$$

*Proof.* In analogy to the construction of the set  $S$  in the proof of Lemma 4.2, starting with  $\partial Q(\gamma)$  we may construct a set  $U$  with  $9 < \text{dist}(x, \gamma) < 11$  for all  $x \in \partial U$  and such that  $\partial U$  is transversal to all  $i$ -cells and has a tubular neighborhood of radius 3. Given an  $M$  in  $C_{\mathcal{A}}^j(\gamma)$ , we may now construct an element  $M'$  in  $C_{\mathcal{A}}^j(\gamma = \emptyset)$ , which agrees with  $M$  outside the set  $U$ . Conversely to any  $M$  in  $C_{\mathcal{A}}^j(\gamma = \emptyset)$ , we first construct an element  $M''$  in  $C_{\mathcal{A}}^j(\gamma = \emptyset)$ , which agrees with  $M$  outside of  $U$  and such that  $\text{dist}(x, \gamma) > 3$  for  $x \in M''$ . Taking the union of  $M''$  with the minimal area for  $\gamma$  gives an element  $M'''$  in  $C_{\mathcal{A}}^j(\gamma)$ . The estimate (4.28) is now derived by the same arguments as used in the proof of Lemma 4.2. ■

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